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COMMENT

Comment on 'Replica analysis of the p-spin interaction Ising spin-glass model'

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Abstract

We demonstrate that the analytic calculation of the 1RSB break point parameter in a paper by de Oliveira and Fontanari (de Oliveira V M and Fontanari J F 1998 *J. Phys. A: Math. Gen.* **32** 2285) is erroneous, due to the omission of a higher-order term in a lengthy perturbative calculation, and provide a refinement of the accompanying numerical results.

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In 1999, de Oliveira and Fontanari (OF) studied the one-step replica symmetry breaking (1RSB) of a glass of Ising spins with a quenched random p-spin interaction of infinite range in a field [1]. The Hamiltonian is given by

$$\mathcal{H} = \sum_{i_1<i_2\cdots<i_p} J_{i_1,\ldots,i_p}\sigma_{i_1}\cdots\sigma_{i_p} - h\sum_i \sigma_i \tag{1}$$

where the $J_{i_1,\ldots,i_p}$ are independent Gaussian variables with zero mean and variance $p!J^2/2N^{p-1}$. They found that for fields h less than a critical value h_c the transition was discontinuous (D1RSB), while for $h > h_c$ it is continuous (C1RSB). In section 3.1 of that paper, they give certain results on the C1RSB line, including an expression for the 1RSB break point quantity x (m in their notation). We demonstrate an error in this calculation of x. Moreover, they present numerical results within the 1RSB phase which are (for reasons we shall explain below) inaccurate near the C1RSB line. We present a discussion and refined results.

The self-consistent equation for the RS phase is [2]

$$q = T(2) \tag{2}$$

where

$$T(n) \doteq \int \frac{\mathrm{d}z}{\sqrt{2\pi}}\,\mathrm{e}^{-z^2/2}\tanh^n\left(\sqrt{\frac{1}{2}p\beta^2J^2q^{p-1}}\,z + \beta h\right). \tag{3}$$

This solution becomes unstable against small replica symmetry breaking fluctuations on the Almeida–Thouless line, given by [2]

$$kS(4) = 1 \tag{4}$$

where

$$k \doteq \frac{1}{2}p(p-1)\beta^2 J^2 q^{p-2} \tag{5}$$

$$S(n) \doteq \int \frac{\mathrm{d}z}{\sqrt{2\pi}}\,\mathrm{e}^{-z^2/2}\mathrm{sech}^n\left(\sqrt{\frac{1}{2}p\beta^2 J^2 q^{p-1}}\,z + \beta h\right). \tag{6}$$

The self-consistent equations for the 1RSB phase are [1]

$$q_0 = \int \frac{\mathrm{d}z_0}{\sqrt{2\pi}}\,\mathrm{e}^{-z_0^2/2}\left(\frac{\int \frac{\mathrm{d}z_1}{\sqrt{2\pi}}\,\mathrm{e}^{-z_1^2/2}\cosh^x G\,\tanh G}{\int \frac{\mathrm{d}z_1}{\sqrt{2\pi}}\,\mathrm{e}^{-z_1^2/2}\cosh^x G}\right)^2 = 0 \tag{7a}$$

$$q_1 = \int \frac{\mathrm{d}z_0}{\sqrt{2\pi}}\,\mathrm{e}^{-z_0^2/2}\,\frac{\int \frac{\mathrm{d}z_1}{\sqrt{2\pi}}\,\mathrm{e}^{-z_1^2/2}\cosh^x G\,\tanh^2 G}{\int \frac{\mathrm{d}z_1}{\sqrt{2\pi}}\,\mathrm{e}^{-z_1^2/2}\cosh^x G} = 0 \tag{7b}$$

$$\begin{aligned}\frac{1}{4}(p-1)\beta^2 J^2(q_1^p - q_0^p) = -\frac{1}{x^2}\int \frac{\mathrm{d}z_0}{\sqrt{2\pi}}\,\mathrm{e}^{-z_0^2/2}\,\ln\int \frac{\mathrm{d}z_1}{\sqrt{2\pi}}\,\mathrm{e}^{-z_1^2/2}\cosh^x G\\ +\frac{1}{x}\int \frac{\mathrm{d}z_0}{\sqrt{2\pi}}\,\mathrm{e}^{-z_0^2/2}\,\frac{\int \frac{\mathrm{d}z_1}{\sqrt{2\pi}}\,\mathrm{e}^{-z_1^2/2}\cosh^x G\,\ln\cosh G}{\int \frac{\mathrm{d}z_1}{\sqrt{2\pi}}\,\mathrm{e}^{-z_1^2/2}\cosh^x G} = 0\end{aligned} \tag{7c}$$

where

$$G \doteq \sqrt{\frac{1}{2}p\beta^2 J^2 q_0^{p-1}}\,z_0 + \sqrt{\frac{1}{2}p\beta^2 J^2 (q_1^{p-1} - q_0^{p-1})}\,z_1 + \beta h\,. \tag{8}$$

The C1RSB line is defined by $q_0 = q_1$. In this case, G does not depend on z_1, and the integrals over this variable are trivial. We observe that (7a) and (7b) reduce to the same equation, namely (3): on this line, the solution coincides with the RS, as we would expect. We obtain a second piece of information by subtracting these equations and performing a series expansion in the quantity $\epsilon = q_1 - q_0$: (7a) and (7b) both become

$$q = T(2) + O(\epsilon) \tag{9}$$

and the difference (7b) − (7a) becomes

$$\epsilon = kS(4)\epsilon + O(\epsilon)^2. \tag{10}$$

So (4) is also satisfied on the C1RSB line; that is, the transition coincides with the onset of instability in the RS solution, again as we would expect.

On the C1RSB line, (7c) is trivially solved. We obtain further information by a series expansion. To first order we get

$$\frac{1}{2}kq\epsilon = \frac{1}{2}kT(2)\epsilon + O(\epsilon)^2. \tag{11}$$

This simply tells us that (3) holds, which we already knew. We therefore eliminate this first order term by subtracting (7a) multiplied by $k\epsilon/2$ from (7c) to obtain a new equation. To second order we get, rearranging slightly,

$$\frac{1}{4}k\left(\frac{p-2}{q}[q - T(2)] + [1 - kS(4)]\right)\epsilon^2 + O(\epsilon)^3 = 0. \tag{12}$$

This tells us that (4) holds, which again we already knew. We therefore eliminate these second order terms by subtracting (7a) multiplied by $(p-2)k\epsilon^2/4q$ and [(7b) − (7a)] multiplied by $k\epsilon/4$ to obtain another new equation. To third order we get

$$\frac{k^2}{24q}\left[\frac{p-2}{k}C + 4qkS(4) - 6qkS(6) + 2qkS(6)(1-x)\right]\epsilon^3 + O(\epsilon)^4 = 0 \tag{13}$$

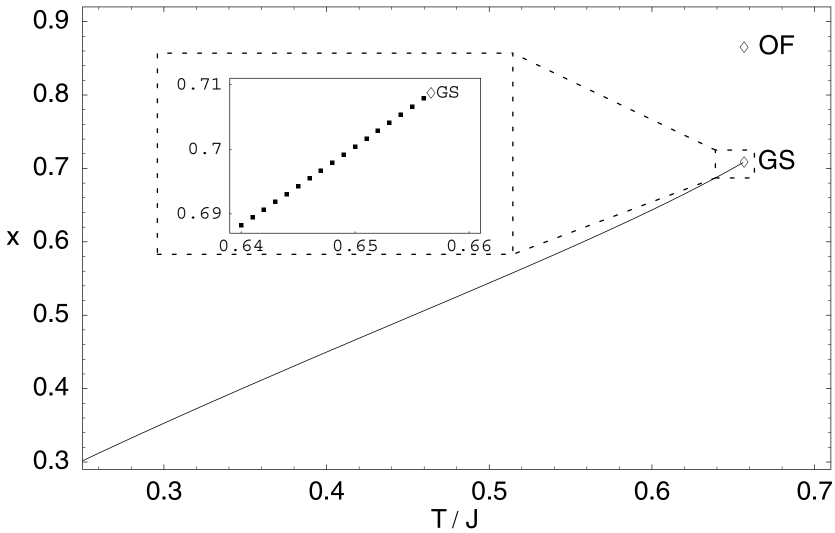

Figure 1. The numerical solution of (7) for x with $p = 3$ and $h/J = 1$, and a comparison with the analytic results at the transition.

where

$$C \doteq 2(p-1) - 2(p-3)\frac{T(2)}{q} - 3kS(4). \tag{14}$$

We solve this to obtain an expression for x near the C1RSB line:

$$1 - x = \frac{6qkS(6) - 4qkS(4) - (p-2)Ck^{-1}}{2kqS(6)} + O(\epsilon) \tag{15}$$

$$= \frac{6qkS(6) - [4qk + (p-2)]S(4)}{2kqS(6)} + O(\epsilon) \tag{16}$$

where we have used (9) and (10) to simplify our expression. This differs from (36) and surrounding equations of OF. We note that if one erroneously neglects the $O(\epsilon)$ terms of (7a) when subtracting that equation multiplied by $(p-2)k\epsilon^2/4q$ in the above process (using only the leading-order equation (3) instead) one obtains an incorrect form of the $O(\epsilon)^3$ equation (13) which gives exactly the form of OF.

Since we have shown the analytic expression of OF for x near the C1RSB line to be incorrect, we must question the accuracy of their numerical solutions of the 1RSB equations (7) in that region, as the latter appeared to corroborate the former. The determination of x is indeed rather delicate, as the x-dependence of these equations is very weak, for reasons that are clear from the above analysis: it appears in a term $O(q_1 - q_0)^2$ smaller than the leading order, and close to C1RSB, $(q_1 - q_0) \ll 1$ by definition.

We adopt an approach designed to avoid this problem. Rather than solving the equations as given, we choose a judicious linear combination which does not possess the same flatness. We know from above that subtracting (7a) multiplied by $k(q_1 - q_0)/2$ from (7c) eliminates the leading order of the latter, leaving an equation where the x-dependence is suppressed only by a factor $O(q_1 - q_0)$; and that further subtracting (7a) multiplied by $(p-2)k(q_1 - q_0)^2/4q$ and [(7b) − (7a)] multiplied by $k(q_1 - q_0)/4$ eliminates the next order, leaving an equation whose leading order is linear in x. We find it most efficient to use the second of these very close to C1RSB (where the problem is worst) and the first elsewhere. We use a modified form of Newton–Raphson to find the roots, and do the numerical integration using Gauss–Hermite quadrature.

Figure 1 shows the solution at $p = 3$ and $h/J = 1$ as a function of temperature. This is equivalent to the solid line in figure 3 of OF. The main line shows the numerical solution for x. The two diamonds show the predictions for x on the C1RSB line, the lower using (16) and the upper using the equivalent expression of OF. The inset shows an enlargement of the region around the transition, with rectangles for the numerical predictions for x and a diamond for our perturbative result.

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